

A CONSTRUCTIVE APPROACH TO THE FINITE WAVELET FRAMES OVER PRIME FIELDS

ASGHAR RAHIMI AND NILOUFAR SEDDIGHI

ABSTRACT. In this article, we present a constructive method for computing the frame coefficients of finite wavelet frames over prime fields using tools from computational harmonic analysis and group theory.

Keywords: Finite wavelet frames, finite wavelet group, prime fields.

MSC(2010): Primary 42C15, 42C40, 65T60; Secondary 30E05, 30E10.

1. Introduction

The mathematical theory of frames in Hilbert spaces was introduced in [10], and has been studied in depth for finite dimensions in [5, 7]. Finite frames have been applied as means to interpret periodic signals and privileged in areas ranging from digital signal processing to image analysis, filter banks, big data and compressed sensing, see [7, 28, 29] and references therein. The representations of a function/signal in time-frequency (resp. time-scale) domain are obtained through analyzing the signal with respect to an over complete system whose elements are localized in time and frequency (resp. scale) [3, 4, 19, 20].

In the framework of wave packet analysis [8], finite wavelet systems are particular classes of finite wave packet systems which have been introduced recently in [15, 16, 17]. The analytic structure of finite wavelet frames over prime fields (finite Abelian groups of prime orders) has been studied in [14, 21]. The notion of a wavelet transform over a prime field was introduced in [12] and extended for finite fields in [11, 23]. This notion is an example of the abstract generalization of wavelet transforms using harmonic analysis, see [1, 2, 6, 9, 13, 28] and classical references therein. The current paper consists of a constructive approach to the abstract aspects of nature of finite wavelet systems over prime fields. The motivation of this paper is to establish an alternative constructive formulation for the wavelet coefficients of finite wavelet frames over prime fields.

This article is organized as follows. Section 2 introduces some notations as well as a brief review of Fourier transform on finite cyclic groups, periodic signal processing, and finite frames. Then in section 3 we present a constructive technique for computing the frame coefficients of finite wavelet frames over prime fields using tools from computational harmonic analysis and theoretical group theory. In addition, we shall also give a constructive

characterization for frame conditions of finite wavelet systems over prime fields using matrix analysis terminology.

2. Preliminaries

This section is devoted to present a brief review of notations, basics, and preliminaries of Fourier analysis and computational harmonic analysis over finite cyclic groups, for any details we refer the readers to [25] and classical references therein. It should be mentioned that in this article p is a positive prime integer. We also employ notations of the author of the references [14, 15, 16].

For a finite group G , the complex vector space $\mathbb{C}^G = \{\mathbf{x} : G \rightarrow \mathbb{C}\}$ is a $|G|$ -dimensional vector space with complex vector entries indexed by elements in the finite group G . The inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^G$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{g \in G} \mathbf{x}(g) \overline{\mathbf{y}(g)}$, and the induced norm is the $\|\cdot\|_2$ -norm of \mathbf{x} . For $\mathbb{C}^{\mathbb{Z}_p}$, where \mathbb{Z}_p denotes the cyclic group of p elements $\{0, \dots, p-1\}$, we write \mathbb{C}^p . The notation $\|\mathbf{x}\|_0 = |\{k \in \mathbb{Z}_p : \mathbf{x}(k) \neq 0\}|$ counts non-zero entries in $\mathbf{x} \in \mathbb{C}^p$. The translation operator $T_k : \mathbb{C}^p \rightarrow \mathbb{C}^p$ is $T_k \mathbf{x}(s) = \mathbf{x}(s - k)$ for $\mathbf{x} \in \mathbb{C}^p$ and $k, s \in \mathbb{Z}_p$. The modulation operator $M_\ell : \mathbb{C}^p \rightarrow \mathbb{C}^p$ is given by $M_\ell \mathbf{x}(s) = e^{-2\pi i \ell s/p} \mathbf{x}(s)$ for $\mathbf{x} \in \mathbb{C}^p$ and $\ell, s \in \mathbb{Z}_p$. The translation and modulation operators on the Hilbert space \mathbb{C}^p are unitary operators in the $\|\cdot\|_2$ -norm. For $\ell, k \in \mathbb{Z}_p$ we have $(T_k)^* = (T_k)^{-1} = T_{p-k}$ and $(M_\ell)^* = (M_\ell)^{-1} = M_{p-\ell}$. The unitary discrete Fourier Transform (DFT) of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^p$ is defined by $\widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(k) \overline{\mathbf{w}_\ell(k)}$, for all $\ell \in \mathbb{Z}_p$, where for all $\ell, k \in \mathbb{Z}_p$ we have $\mathbf{w}_\ell(k) = e^{2\pi i \ell k/p}$. Thus, DFT of $\mathbf{x} \in \mathbb{C}^p$ at $\ell \in \mathbb{Z}_p$ is

$$(2.1) \quad \widehat{\mathbf{x}}(\ell) = \mathcal{F}_p(\mathbf{x})(\ell) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(k) \overline{\mathbf{w}_\ell(k)} = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(k) e^{-2\pi i \ell k/p}.$$

The DFT is a unitary transform in $\|\cdot\|_2$ -norm, i.e. for all $\mathbf{x} \in \mathbb{C}^p$ satisfies the Parseval formula $\|\mathcal{F}_p(\mathbf{x})\|_2 = \|\mathbf{x}\|_2$. The Polarization identity implies the Plancherel formula $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$. The unitary DFT (2.1) satisfies $\widehat{T_k \mathbf{x}} = M_k \widehat{\mathbf{x}}$, $\widehat{M_\ell \mathbf{x}} = T_{p-\ell} \widehat{\mathbf{x}}$, for $\mathbf{x} \in \mathbb{C}^p$ and $k, \ell \in \mathbb{Z}_p$. Also the inverse discrete Fourier formula (IDFT) for a 1D discrete signal $\mathbf{x} \in \mathbb{C}^p$ is given by

$$\mathbf{x}(k) = \frac{1}{\sqrt{p}} \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell) \mathbf{w}_\ell(k) = \frac{1}{\sqrt{p}} \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell) e^{2\pi i \ell k/p}, \quad 0 \leq k \leq p-1.$$

A finite sequence $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\} \subset \mathbb{C}^p$ is called a frame (or finite frame) for \mathbb{C}^p , if there exists a positive constant $0 < A < \infty$ such that [7]

$$A \|\mathbf{x}\|_2^2 \leq \sum_{j=0}^{M-1} |\langle \mathbf{x}, \mathbf{y}_j \rangle|^2, \quad (\mathbf{x} \in \mathbb{C}^p).$$

The synthesis operator $F : \mathbb{C}^M \rightarrow \mathbb{C}^p$ is defined by $F\{c_j\}_{j=0}^{M-1} = \sum_{j=0}^{M-1} c_j \mathbf{y}_j$ for all $\{c_j\}_{j=0}^{M-1} \in \mathbb{C}^M$. The adjoint operator $F^* : \mathbb{C}^p \rightarrow \mathbb{C}^M$ is defined by $F^* \mathbf{x} = \{\langle \mathbf{x}, \mathbf{y}_j \rangle\}_{j=0}^{M-1}$ for all $\mathbf{x} \in \mathbb{C}^p$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a frame for \mathbb{C}^p , by composing F and F^* , we get the positive and invertible frame operator $S : \mathbb{C}^p \rightarrow \mathbb{C}^p$ given by

$$\mathbf{x} \mapsto S\mathbf{x} = FF^* \mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle \mathbf{y}_j, \quad (\mathbf{x} \in \mathbb{C}^p),$$

and the set \mathfrak{A} spans the complex Hilbert space \mathbb{C}^p which implies $M \geq p$, where $M = |\mathfrak{A}|$. Each finite spanning set in \mathbb{C}^p is a finite frame for \mathbb{C}^p . The ratio between M and p is called the redundancy of the finite frame \mathfrak{A} (i.e. $\text{red}_{\mathfrak{A}} = M/p$). If \mathfrak{A} is a finite frame for \mathbb{C}^p , each $\mathbf{x} \in \mathbb{C}^p$ satisfies

$$\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, S^{-1} \mathbf{y}_j \rangle \mathbf{y}_j = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle S^{-1} \mathbf{y}_j.$$

3. Construction of Wavelet Frames over Prime Fields

In this section, we briefly state structure and basic properties of cyclic dilation operators, see [12, 14, 18, 23]. Then we present an overview for the notion and structure of finite wavelet groups over prime fields [24, 26].

3.1. Structure of Finite Wavelet Group over Prime Fields. The set

$$\mathbb{U}_p := \mathbb{Z}_p - \{0\} = \{1, \dots, p-1\},$$

is a finite multiplicative Abelian group of order $p-1$ with respect to the multiplication module p with the identity element 1. The multiplicative inverse for $m \in \mathbb{U}_p$ is m_p which satisfies $m_p m + np = 1$ for some $n \in \mathbb{Z}$, see [22, 27].

For $m \in \mathbb{U}_p$, the cyclic dilation operator is defined by $D_m : \mathbb{C}^p \rightarrow \mathbb{C}^p$ by

$$D_m \mathbf{x}(k) := \mathbf{x}(m_p k)$$

for $\mathbf{x} \in \mathbb{C}^p$ and $k \in \mathbb{Z}_p$, where m_p is the multiplicative inverse of m in \mathbb{U}_p .

In the following propositions, we state some basic properties of cyclic dilations.

Proposition 3.1. *Let p be a positive prime integer. Then*

- (i) *For $(m, k) \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $D_m T_k = T_{mk} D_m$.*
- (ii) *For $m, m' \in \mathbb{U}_p$, we have $D_{mm'} = D_m D_{m'}$.*
- (iii) *For $(m, k), (m', k') \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $T_{k+mk'} D_{mm'} = T_k D_m T_{k'} D_{m'}$.*
- (iv) *For $(m, \ell) \in \mathbb{U}_p \times \mathbb{Z}_p$, we have $D_m M_\ell = M_{m_p \ell} D_m$.*

The next result states some properties of cyclic dilations.

Proposition 3.2. *Let p be a positive prime integer and $m \in \mathbb{U}_p$. Then*

- (i) *The dilation operator $D_m : \mathbb{C}^p \rightarrow \mathbb{C}^p$ is a $*$ -homomorphism.*

- (ii) The dilation operator $D_m : \mathbb{C}^p \rightarrow \mathbb{C}^p$ is unitary in $\|\cdot\|_2$ -norm and satisfies

$$(D_m)^* = (D_m)^{-1} = D_{m_p}.$$

- (iii) For $\mathbf{x} \in \mathbb{C}^p$, we have $\widehat{D_m \mathbf{x}} = D_{m_p} \widehat{\mathbf{x}}$.

The underlying set

$$\mathbb{U}_p \times \mathbb{Z}_p = \{(m, k) : m \in \mathbb{U}_p, k \in \mathbb{Z}_p\},$$

equipped with the following group operations

$$(m, k) \times (m', k') := (mm', k + mk'),$$

$$(m, k)^{-1} := (m_p, m_p \cdot (p - k)),$$

is a finite non-Abelian group of order $p(p-1)$ denoted by \mathbb{W}_p and it is called as finite affine group on p integers or the finite wavelet group associated to the integer p or simply as p -wavelet group.

Next proposition states basic properties of the finite wavelet group \mathbb{W}_p .

Proposition 3.3. *Let $p > 2$ be a positive prime integer. Then \mathbb{W}_p is a non-Abelian group of order $p(p-1)$ which contains a copy of \mathbb{Z}_p as a normal Abelian subgroup and a copy of \mathbb{U}_p as a non-normal Abelian subgroup.*

3.2. Wavelet Frames over Prime Fields. A finite wavelet system for the complex Hilbert space \mathbb{C}^p is a family or system of the form

$$\mathcal{W}(\mathbf{y}, \Delta) := \{\sigma(m, k)\mathbf{y} = T_k D_m \mathbf{y} : (m, k) \in \Delta \subseteq \mathbb{W}_p\},$$

for some window signal $\mathbf{y} \in \mathbb{C}^p$ and a subset Δ of \mathbb{W}_p .

If $\Delta = \mathbb{W}_p$, we put $\mathcal{W}(\mathbf{y}) := \mathcal{W}(\mathbf{y}, \mathbb{W}_p)$ and it is called a full finite wavelet system over \mathbb{Z}_p . A finite wavelet system which spans \mathbb{C}^p is a frame and is referred to as a finite wavelet frame over the prime field \mathbb{Z}_p .

If $\mathbf{y} \in \mathbb{C}^p$ is a window signal then for $\mathbf{x} \in \mathbb{C}^p$, we have

$$\langle \mathbf{x}, \sigma(m, k)\mathbf{y} \rangle = \langle \mathbf{x}, T_k D_m \mathbf{y} \rangle = \langle T_{p-k} \mathbf{x}, D_m \mathbf{y} \rangle, \quad ((m, k) \in \mathbb{W}_p).$$

The following proposition states a formulation for wavelet coefficients via Fourier transform.

Proposition 3.4. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ and $(m, k) \in \mathbb{W}_p$. Then,*

$$\langle \mathbf{x}, \sigma(m, k)\mathbf{y} \rangle = \sqrt{p} \mathcal{F}_p(\widehat{\mathbf{x}} \cdot \widehat{D_m \mathbf{y}})(p - k).$$

Proof. See Proposition 4.1 of [14]. □

In [14] using an analytic approach the author has presented a concrete formulation for the $\|\cdot\|_2$ -norm of wavelet coefficients, the formula of which is just stated hereby.

Theorem 3.5. *Let p be a positive prime integer, M a divisor of $p - 1$ and \mathbb{M} be a multiplicative subgroup of \mathbb{U}_p of order M . Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal and $\mathbf{x} \in \mathbb{C}^p$. Then,*

$$\begin{aligned} & \sum_{m \in \mathbb{M}} \sum_{k \in \mathbb{Z}_p} |\langle \mathbf{x}, \sigma(m, k) \mathbf{y} \rangle|^2 \\ &= p \left(M |\widehat{\mathbf{y}}(0)|^2 |\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m)|^2 \right) \left(\sum_{\ell \in \mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2 \right) + \sum_{\ell \in \mathbb{U}_p - \mathbb{M}} \gamma_\ell(\mathbf{y}, \mathbb{M}) |\widehat{\mathbf{x}}(\ell)|^2 \right), \end{aligned}$$

where

$$\gamma_\ell(\mathbf{y}, \mathbb{M}) := \sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m\ell)|^2, \quad (\ell \in \mathbb{U}_p - \mathbb{M}).$$

Proof. See Theorem 4.2 of [14]. \square

Remark 3.6. The formulation presented in Theorem 3.5 is an analytic formulation of wavelet coefficients associated to the subgroup \mathbb{M} . In details, that formulation originated from an analytic approach which was based on direct computations of cyclic dilations in the subgroup \mathbb{M} .

In the following theorem, we present a constructive formulation for the $\|\cdot\|_2$ -norm of wavelet coefficients. At the first, we need to prove some results.

Proposition 3.7. *Let p be a positive prime integer, M a divisor of $p - 1$ and \mathbb{M} a multiplicative subgroup of \mathbb{U}_p of order M . Let ϵ be a generator of the cyclic group \mathbb{U}_p and $a := \frac{p-1}{M}$ the index of \mathbb{M} in \mathbb{U}_p . Then*

- (i) *For $0 \leq r, s \leq a - 1$, we have $\epsilon^r \mathbb{M} = \epsilon^s \mathbb{M}$ iff $r = s$.*
- (ii) *$\mathbb{U}_p / \mathbb{M} = \{\epsilon^t \mathbb{M} : 0 \leq t \leq a - 1\}$.*

Proof. (i) First, suppose $\epsilon^r \mathbb{M} = \epsilon^s \mathbb{M}$ for some $0 \leq r, s \leq a - 1$. We will show $r = s$. For see this, assume to the contrary that $r \neq s$. Without loss of generality, suppose $s > r$. Thus, there exists

$$(3.1) \quad 1 \leq r' \leq a - 1 < a,$$

such that $s = r + r'$, and we get $\epsilon^r \mathbb{M} = \epsilon^{r+r'} \mathbb{M} = \epsilon^r \epsilon^{r'} \mathbb{M}$. Since ϵ^r is an invertible element of \mathbb{U}_p , so $\mathbb{M} = \epsilon^{r'} \mathbb{M}$ and by this $\epsilon^{r'} \in \mathbb{M}$. Also, $\epsilon^{p-1} \stackrel{p}{\equiv} (\epsilon^{r'})^{\frac{p-1}{r'}} \stackrel{p}{\equiv} 1$. Hence the order of $\epsilon^{r'}$ in \mathbb{U}_p can be at most $\frac{p-1}{r'}$. Since ϵ is a generator of \mathbb{U}_p , the order of $\epsilon^{r'}$ is $\frac{p-1}{r'}$. In addition, by (3.1) we get $M < \frac{p-1}{r'} \leq p - 1$, therefore $\epsilon^{r'}$ can not be in \mathbb{M} which is a contradiction. The inverse implication is straightforward.

(ii) According to (i), $\{\epsilon^t \mathbb{M} : 0 \leq t \leq a - 1\}$ is a set of disjoint cosets of \mathbb{M} in \mathbb{U}_p and the cardinality of this set is equal to the index number of \mathbb{M} in \mathbb{U}_p , so the equality in (ii) holds. In particular, this set of cosets creates a partition to \mathbb{U}_p . \square

Next we present a constructive formula for $\|\cdot\|_2$ of wavelet coefficients.

Theorem 3.8. *Let p be a positive prime integer, M a divisor of $p-1$ and \mathbb{M} a multiplicative subgroup of \mathbb{U}_p of order M . Let ϵ be a generator of the cyclic group \mathbb{U}_p and $a := \frac{p-1}{M}$ the index of \mathbb{M} in \mathbb{U}_p . Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal and $\mathbf{x} \in \mathbb{C}^p$. Then,*

$$\begin{aligned} & \sum_{m \in \mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 \\ &= p \left(M |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 + \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right) \right), \end{aligned}$$

where $H_t := \epsilon^t \mathbb{M}$ for all $0 \leq t \leq a-1$.

Proof. Let $\mathbf{y} \in \mathbb{C}^p$ be a window function, $\mathbf{x} \in \mathbb{C}^p$ and $m \in \mathbb{M}$. Then using Proposition 3.4 and Plancherel formula we have

$$\sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 = p \sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \cdot |\widehat{\mathbf{y}}(m\ell)|^2.$$

Hence we achieve

$$\begin{aligned} \sum_{m \in \mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 &= \sum_{m \in \mathbb{M}} p \sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \cdot |\widehat{\mathbf{y}}(m\ell)|^2 \\ &= p \sum_{\ell=0}^{p-1} \sum_{m \in \mathbb{M}} |\widehat{\mathbf{x}}(\ell)|^2 \cdot |\widehat{\mathbf{y}}(m\ell)|^2. \end{aligned}$$

Then we have

$$(3.2) \quad \sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m\ell)|^2 \right) = |\widehat{\mathbf{x}}(0)|^2 \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(0)|^2 \right) + \sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m\ell)|^2 \right).$$

Now by (3.2) and Proposition 3.7, we get

$$\begin{aligned} \sum_{m \in \mathbb{M}} \sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 |\widehat{\mathbf{y}}(m\ell)|^2 &= M |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 \\ &\quad + \sum_{t=0}^{a-1} \sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m\ell)|^2 \right). \end{aligned}$$

Replacing $m\ell$ with w we get, $w \stackrel{p}{\equiv} m\ell \in \mathbb{M} \cdot H_t = \mathbb{M} \cdot \epsilon^t \mathbb{M} = \epsilon^t \mathbb{M} = H_t$. For each $\ell \in H_t$, the mapping $m \rightarrow m\ell$ defines a bijection of M onto H_t , and

thus we deduce that

$$\begin{aligned} \sum_{m \in \mathbb{M}} \sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 |\widehat{\mathbf{y}}(m\ell)|^2 &= M |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 \\ &+ \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{m \in \mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 \\ = p \left(M |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 + \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right) \right). \end{aligned}$$

□

The following result is an immediate consequence of Theorem 3.8.

Corollary 3.9. *Let p be a positive prime integer. Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal and $\mathbf{x} \in \mathbb{C}^p$. Then*

$$\begin{aligned} \sum_{m \in \mathbb{U}_p} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 \\ = p \left((p-1) |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 + \left(\sum_{\ell \in \mathbb{U}_p} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in \mathbb{U}_p} |\widehat{\mathbf{y}}(w)|^2 \right) \right). \end{aligned}$$

Applying Theorem 3.8, we can present the following constructive characterization of a given window signal $\mathbf{y} \in \mathbb{C}^p$ and a subgroup of the finite wavelet group \mathbb{W}_p to guarantee that generated wavelet system is a frame for \mathbb{C}^p .

Theorem 3.10. *Let p be a positive prime integer, ϵ a generator of \mathbb{U}_p , M a divisor of $p-1$, \mathbb{M} a multiplicative subgroup of \mathbb{U}_p of order M and $a := \frac{p-1}{M}$ the index of \mathbb{M} in \mathbb{U}_p . Let $\Delta_{\mathbb{M}} := \mathbb{M} \times \mathbb{Z}_p$ and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for \mathbb{C}^p if and only if the following conditions hold*

- (i) $\widehat{\mathbf{y}}(0) \neq 0$
- (ii) For each $t \in \{0, \dots, a-1\}$, there exists $m_t \in \mathbb{M}$ such that $\widehat{\mathbf{y}}(\epsilon^t m_t) \neq 0$.

Proof. Let \mathbf{y} be a non-zero window signal which satisfies conditions (i), (ii). Then by definition of H_t we get

$$\vartheta := \min_{t \in \{0, \dots, a-1\}} \left(\sum_{m \in \mathbb{M}} |\widehat{\mathbf{y}}(m\epsilon^t)|^2 \right) = \min_{t \in \{0, \dots, a-1\}} \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right) \neq 0.$$

Now, let $0 < A < \infty$ be given by

$$A := \min \left\{ M \left| \sum_{k=0}^{p-1} \mathbf{y}(k) \right|^2, p\vartheta \right\}.$$

Using Theorem 3.8 for $\mathbf{x} \in \mathbb{C}^p$, we have

$$\begin{aligned} \sum_{m \in \mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 &= pM |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 + p \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right) \\ &\geq pM |\widehat{\mathbf{x}}(0)|^2 |\widehat{\mathbf{y}}(0)|^2 + p\vartheta \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \\ &\geq A |\widehat{\mathbf{x}}(0)|^2 + A \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \\ &= A |\widehat{\mathbf{x}}(0)|^2 + A \sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \\ &= A \|\widehat{\mathbf{x}}\|_2^2 = A \|\mathbf{x}\|_2^2. \end{aligned}$$

which implies that the finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for \mathbb{C}^p .

For the inverse implication, consider $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal such that the finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for \mathbb{C}^p . Thus, there exists $A_1 > 0$ such that

$$\sum_{m \in \mathbb{M}} \sum_{k=0}^{p-1} |\langle \mathbf{x}, T_k D_m \mathbf{y} \rangle|^2 \geq A_1 \|\mathbf{x}\|_2^2, \quad (\mathbf{x} \in \mathbb{C}^p).$$

Then by Theorem 3.8 we have

$$(3.3) \quad M |\widehat{\mathbf{y}}(0)|^2 |\widehat{\mathbf{x}}(0)|^2 + \sum_{t=0}^{a-1} \left(\sum_{\ell \in H_t} |\widehat{\mathbf{x}}(\ell)|^2 \right) \left(\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2 \right) \geq A_2 \|\mathbf{x}\|_2^2$$

for all $\mathbf{x} \in \mathbb{C}^p$, where $A_2 = \frac{A_1}{p}$. Now let $\mathbf{x}' \in \mathbb{C}^p$ with $\widehat{\mathbf{x}}'(0) \neq 0$ and $\widehat{\mathbf{x}}'(\ell) = 0$, for all $\ell \in \{1, \dots, p-1\}$. Thus, by (3.3) we get $\widehat{\mathbf{y}}(0) \neq 0$. Next, consider $\mathbf{x}_t \in \mathbb{C}^p$ be a non-zero vector in which for a fixed but arbitrary $t \in \{0, \dots, a-1\}$,

$$(3.4) \quad \widehat{\mathbf{x}}_t(\ell) = 0, \forall \ell \notin H_t.$$

Hence (3.4) assures that $\sum_{w \in H_t} |\widehat{\mathbf{y}}(w)|^2$ should be non-zero. Therefore $\widehat{\mathbf{y}}(w) \neq 0$ for some $w \in H_t$, which indicates that there exists $m_t \in \mathbb{M}$ such that $\widehat{\mathbf{y}}(m_t \epsilon^t) \neq 0$. \square

The following result shows that for a large class of non-zero window signals the finite wavelet system $\mathcal{W}(\mathbf{y})$ is a frame for \mathbb{C}^p with redundancy $p-1$.

Corollary 3.11. *Let p be a positive prime integer and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y})$ constitutes a frame for \mathbb{C}^p with the redundancy $p - 1$ if and only if $\hat{\mathbf{y}}(0) \neq 0$ and $\|\hat{\mathbf{y}}\|_0 \geq 2$.*

The following corollary presents a characterization for finite wavelet systems over prime fields using matrix analysis language.

Corollary 3.12. *Let p be a positive prime integer, ϵ a generator of \mathbb{U}_p , M a divisor of $p - 1$, \mathbb{M} a multiplicative subgroup of \mathbb{U}_p of order M and $a := \frac{p-1}{M}$ be the index of \mathbb{M} in \mathbb{U}_p . Let $\Delta_{\mathbb{M}} := \mathbb{M} \times \mathbb{Z}_p$ and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a frame for \mathbb{C}^p if and only if $\hat{\mathbf{y}}(0) \neq 0$ and the matrix $\mathbf{Y}(\mathbb{M}, \mathbf{y})$ of size $a \times M$ given by*

$$\mathbf{Y}(\mathbb{M}, \mathbf{y}) := \begin{bmatrix} \hat{\mathbf{y}}(1) & \hat{\mathbf{y}}(\epsilon^a) & \dots & \hat{\mathbf{y}}(\epsilon^{(M-1)a}) \\ \hat{\mathbf{y}}(\epsilon^1) & \hat{\mathbf{y}}(\epsilon^{a+1}) & \dots & \hat{\mathbf{y}}(\epsilon^{(M-1)a+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{y}}(\epsilon^{a-1}) & \hat{\mathbf{y}}(\epsilon^{2a-1}) & \dots & \hat{\mathbf{y}}(\epsilon^{Ma-1}) \end{bmatrix}_{a \times M}$$

is a matrix such that each row has at least a non-zero entry.

Proof. Since ϵ^a is a generator of \mathbb{M} and $|\mathbb{M}| = M$, so

$$(3.5) \quad \mathbb{M} = \{(\epsilon^a)^0, (\epsilon^a)^1, \dots, (\epsilon^a)^{M-1}\} = \{1, \epsilon^a, \dots, \epsilon^{(M-1)a}\}.$$

By (3.5) and definition of H_t , we have

$$H_t = \{\epsilon^t, \epsilon^a \cdot \epsilon^t, \dots, \epsilon^{(M-1)a} \cdot \epsilon^t\} = \{\epsilon^t, \epsilon^{a+t}, \dots, \epsilon^{(M-1)a+t}\}$$

for all $t \in \{0, \dots, a-1\}$. The entries of t -th row of $\mathbf{Y}(\mathbb{M}, \mathbf{y})$ are entries of $\hat{\mathbf{y}}$ on H_t . Hence by Theorem 3.10 the result holds. \square

We can also deduce the following tight frame condition for finite wavelet systems generated by subgroups of \mathbb{W}_p .

Proposition 3.13. *Let p be a positive prime integer, M a divisor of $p - 1$ and \mathbb{M} a multiplicative subgroup of \mathbb{U}_p of order M . Let $\Delta_{\mathbb{M}} := \mathbb{M} \times \mathbb{Z}_p$ and $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The finite wavelet system $\mathcal{W}(\mathbf{y}, \Delta_{\mathbb{M}})$ is a tight frame for \mathbb{C}^p if and only if $\hat{\mathbf{y}}(0) \neq 0$ and*

$$M \left| \sum_{k=0}^{p-1} \mathbf{y}(k) \right|^2 = p \left(\sum_{w \in H_t} |\hat{\mathbf{y}}(w)|^2 \right)$$

for all $t \in \{0, \dots, a-1\}$. In this case

$$\alpha_{\mathbf{y}} := pM |\hat{\mathbf{y}}(0)|^2 = p \sum_{m \in \mathbb{M}} |\hat{\mathbf{y}}(m)|^2$$

is the frame bound.

Proof. It can be proven by a similar argument used in Theorem 3.10. \square

Acknowledgement

Some of the results are obtained during the second author's appointment from the NuHAG group at the University of Vienna. We would like to thank Prof. Hans. G. Feichtinger for his valuable comments and the group for their hospitality.

REFERENCES

- [1] A.A. Arefijamaal, R.A. Kamyabi-Gol, *On the square integrability of quasi regular representation on semidirect product groups*, J. Geom. Anal. 19 (3) (2009) 541-552.
- [2] A.A. Arefijamaal, R. Kamyabi-Gol, *On construction of coherent states associated with semidirect products*, Int. J. Wavelets Multiresolut. Inf. Process. 6 (5) (2008) 749-759.
- [3] A.A. Arefijamaal, E. Zekaei, *Signal processing by alternate dual Gabor frames*, Appl. Comput. Harmon. Anal. 35 (2013) 535-540.
- [4] A. A. Arefijamaal, E. Zekaei, *Image processing by alternate dual Gabor frames*, Bull. Iranian Math. Soc. 42 (6) (2016) 1305-1314.
- [5] P. Balazs, *Frames and Finite Dimensionality: Frame Transformation, Classification and Algorithms*, Appl. Math. Sci. 2 (43) (2008) 2131-2144.
- [6] G. Caire, R.L. Grossman, H. Vincent Poor, *Wavelet transforms associated with finite cyclic Groups*, IEEE Trans. Inform. Th. 39 (4) (1993) 1157-1166.
- [7] P. Casazza, G. Kutyniok. *Finite Frames, Theory and Applications*. Appl. and Numer. Harmon. Anal. Birkhäuser, Boston, 2013.
- [8] O. Christensen, A. Rahimi, *Frame properties of wave packet systems in $L^2(\mathbb{R}^d)$* , Adv. Comput. Math. 29 (2) (2008) 101-111.
- [9] I. Daubechies, *Ten Lectures on Wavelets*, (SIAM), Philadelphia, PA, 1992.
- [10] R.J. Duffin, A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Sci. 72 (1952) 341-366.
- [11] F. Fekri, R.M. Mersereau, R.W. Schafer, *Theory of wavelet transform over finite fields*, IEEE. 3 (1999) 1213-1216.
- [12] K. Flornes, A. Grossmann, M. Holschneider and B. Torr sani, *Wavelets on discrete fields*, Appl. Comput. Harmon. Anal. 1 (1994) 137-146.
- [13] H. F hr, *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, Springer-Verlag, 2005.
- [14] A. Ghaani Farashahi, *Structure of finite wavelet frames over prime fields*, Bull. Iranian Math. Soc. 43(1) (2017) 109-120.
- [15] A. Ghaani Farashahi, *Wave packet transform over finite fields*, Electron. J. Linear Algebra, 30 (2015) 507-529.
- [16] A. Ghaani Farashahi, *Wave packet transforms over finite cyclic groups*, Linear Algebra Appl. 489 (2016) 75-92.
- [17] A. Ghaani Farashahi, *Cyclic wavelet systems in prime dimensional linear vector spaces*, Wavelets and Linear Algebra 2 (1) (2015) 11-24.
- [18] A. Ghaani Farashahi, *Cyclic wave packet transform on finite Abelian groups of prime order*, Int. J. Wavelets Multiresolut. Inf. Process. 12 (6) (2014), Article ID 1450041, 14 pages.
- [19] A. Ghaani Farashahi, *Continuous partial Gabor transform for semi-direct product of locally compact groups*, Bull. Malays. Math. Sci. Soc. 38 (2) (2015) 779-803.
- [20] A. Ghaani Farashahi, R. Kamyabi-Gol, *Gabor transform for a class of non-abelian groups*, Bull. Belg. Math. Soc. Simon Stevin 19 (4) (2012) 683-701.

- [21] A. Ghaani Farashahi, M. Mohammad-Pour, *A unified theoretical harmonic analysis approach to the cyclic wavelet transform (CWT) for periodic signals of prime dimensions*, Sahand Commun. Math. Anal. 1 (2) (2014) 1-17.
- [22] G.H. Hardy, E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, New York, (1979).
- [23] C.P. Johnston, *On the pseudodilation representations of flornes, grossmann, holschneider, and torr sani*, J. Fourier Anal. Appl., 3 (4) (1997) 377-385.
- [24] G.L. Mullen, D. Panario, *Handbook of Finite Fields*, Series: Discrete Mathematics and Its Applications, Chapman and Hall/CRC, 2013.
- [25] G. Pfander. *Gabor Frames in Finite Dimensions*, In G.E. Pfander, P.G. Casazza, and G. Kutyniok (eds.), Finite Frames, pp. 193-239, Appl. Numer. Harmon. Anal. Birkh user/Springer, New York, 2013.
- [26] D. Ramakrishnan, R.J. Valenza, *Fourier Analysis on Number Fields*, Springer-Verlag, New York, 1999.
- [27] H. Riesel, *Prime Numbers and Computer Methods for Factorization* (second edition), Boston: Birkhauser, 1994.
- [28] S. Sarkar, H. Vincent Poor, *Cyclic wavelet transforms for arbitrary finite data lengths*, Signal Processing 80 (2000) 2541-2552.
- [29] G. Strang, T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, MA, 1996.

ASGHAR RAHIMI, FACULTY MATHEMATICS, UNIVERSITY OF MARAGHEH, MARAGHEH, IRAN.

E-mail address: rahimi@maragheh.ac.ir

NILOUFAR SEDDIGHI, FACULTY MATHEMATICS, UNIVERSITY OF MARAGHEH, MARAGHEH, IRAN.

E-mail address: stu.seddighi@maragheh.ac.ir